Problem 1.20

Construct a vector function that has zero divergence and zero curl everywhere. (A *constant* will do the job, of course, but make it something a little more interesting than that!)

Solution

In Cartesian coordinates the curl of $\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$ is

$$\begin{aligned}\nabla \times \mathbf{v} &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \\ &= \hat{\mathbf{x}}\left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right),\end{aligned}$$

and the divergence of $\mathbf{v}$ is

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}.$$

Choose, for example,

$$\mathbf{v} = 0\hat{\mathbf{x}} + z\hat{\mathbf{y}} + y\hat{\mathbf{z}}.$$

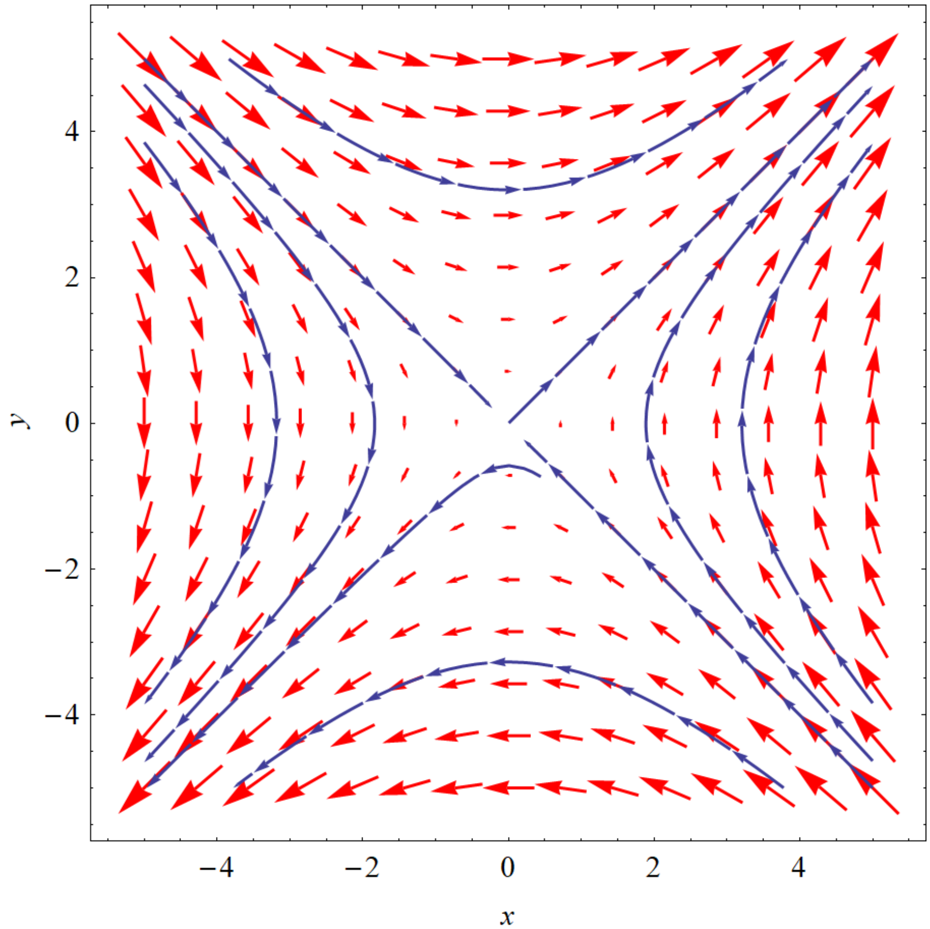
Another vector field with zero divergence and curl is

$$\mathbf{v} = y\hat{\mathbf{x}} + x\hat{\mathbf{y}} + 0\hat{\mathbf{z}}.$$

Another vector field with zero divergence and curl is

$$\mathbf{v} = z\hat{\mathbf{x}} + 0\hat{\mathbf{y}} + x\hat{\mathbf{z}}.$$

$\mathbf{v} = y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$ is illustrated below in red. Some streamlines are drawn in blue.



There's no divergence because for every streamline pointing into the origin, another is pointing out. And there's no curl because, if you imagine a circular wheel centered at the origin, the vector field on the left and right sides turns it counterclockwise; however, the vector field on the top and bottom turns it clockwise. The net result is that the wheel doesn't turn at all.